Int. J. Solids Structures, 1972, Vol. 8, pp. 315 to 326. Pergamon Press. Printed in Great Britain

DYNAMIC BUCKLING OF SOME ELASTIC SHALLOW STRUCTURES SUBJECT TO PERIODIC LOADING WITH HIGH FREQUENCY†

N. C. HUANG

Department of Aerospace and Mechanical Engineering,
University of Notre Dame, Notre Dame, Indiana 46556

Abstract—Dynamic buckling of elastic shallow structures subject to periodic loading is investigated by means of two simple model structures. When the frequency and the magnitude of the oscillatory load are sufficiently high, the cycle averaging technique can be employed to formulate an autonomous system for the cycle-averaged motion of the structure. Energy method is then utilized for determining the upper and lower bounds of the critical load for dynamic buckling.

INTRODUCTION

DYNAMIC buckling of elastic shallow structures has been investigated by many authors [1–7]. In order to include the effect of small initial geometrical imperfections and small disturbances in the study, energy method is usually employed. Upper and lower bounds of the critical load for dynamic buckling can be derived by considering the geometry of the potential surface in a high-dimensional space of generalized coordinates. Although the energy method is a powerful tool for treating dynamic stability problems, its application is, nevertheless, restricted to autonomous systems. Therefore, the energy method can be used successfully to investigate the dynamic buckling of structures under either step loading or impulsive loading. When the applied load is periodic in time, the system is non-stationary. In this case, the energy method cannot be applied directly.

In this paper, we shall consider the dynamic buckling of shallow structures subject to periodic loading by means of two simple model structures: (1) a two-member simple truss with a point mass attached to the middle joint and (2) a slightly imperfect shallow sinusoidal arch supported by two hinges. It is assumed that the frequency of the applied load is sufficiently high that the response of the structure can be regarded as the superposition of a slow motion and a fast oscillation. Dynamic buckling is then investigated by the time variation of the cycle-averaged deformation of the structure. Since the cycle-averaged deformation is autonomous, energy method can be employed in the analysis.

DYNAMIC BUCKLING OF A SIMPLE TRUSS

Let us consider a simple shallow truss of span $2a$ composed by two identical bars connected by frictionless joints as shown in Fig. 1. A point mass M is attached to the middle

† This research was sponsored partly by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant AF-AFOSR 1226-67 with the University of California, San Diego, and partly by the University of Notre Dame.

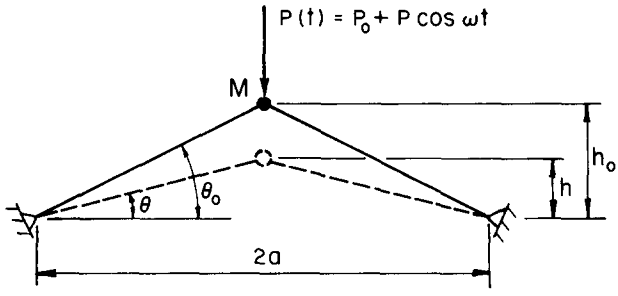

FIG. 1. Geometry of an elastic simple truss.

joint. The initial height of the truss is h_0 and the initial angle of inclination is θ_0. Since the truss is assumed to be shallow, $|\theta_0| \ll 1$. The truss is deformed by a periodic force $P(t) = P_0 + P\cos\omega t$, where P_0 is the mean force, P is the amplitude of the oscillatory force, ω is the frequency of the oscillatory force and t is the time. We shall assume that $P \gg P_0$. The deformed height of the truss at any time is denoted by h and the angle of inclination of the deformed truss is θ. The cross-sectional area of the bar is assumed to be large enough to prevent the bar from lateral buckling during the deformation. Since the truss is shallow, the shortening of the bar at any time t is

$$\Delta = \frac{a}{\cos\theta_0} - \frac{a}{\cos\theta} \simeq a(\cos\theta - \cos\theta_0)$$

$$\simeq \frac{a}{2}(\theta_0^2 - \theta^2) \simeq \frac{1}{2a}(h_0^2 - h^2). \tag{1}$$

The compressive force F in the bar can be found from the equation of motion of the middle joint. It is

$$F = \frac{a}{2h}(M\ddot{h} + P_0 + P\cos\omega t). \tag{2}$$

The bars are assumed to be linearly elastic. Hence Δ is related to F by the following approximate relation:

$$\Delta \simeq \frac{Fa}{AE}, \tag{3}$$

where AE is the extensional stiffness of the bar.

Let us introduce the following dimensionless quantities:

$$Y = \frac{h}{h_0}, \quad p_0 = \frac{P_0 a^3}{AEh_0^2}, \quad p = \frac{Pa^3}{AEh_0^2}, \quad \Omega = \omega\left(\frac{Ma^3}{AEh_0^2}\right)^{1/2}, \quad \tau = t\left(\frac{AEh_0^2}{Ma^3}\right)^{1/2}.$$

After elimination of F and Δ from equations (1)–(3), we obtain the following equation of motion for the shallow truss:

$$\frac{d^2Y}{d\tau^2} + Y^3 - Y + p_0 + p\cos\Omega\tau = 0. \tag{4}$$

Equation (4) defines an instationary nonlinear system with Ω as the frequency of the forcing function. In the following, we shall assume that $\Omega \gg 1$. Put $\xi = \Omega\tau$ and $y = 1 - Y$. We may rewrite equation (4) as

$$\ddot{y} + \frac{1}{\Omega^2}(y^3 - 3y^2 + 2y - p_0 - p\cos\xi) = 0, \tag{5}$$

where dot represents the differentiation with respect to ξ. The general solution of the linearized equation of equation (5) is

$$y = A\cos\frac{(2)^{1/2}}{\Omega}\xi + B\sin\frac{(2)^{1/2}}{\Omega}\xi + \frac{p_0}{2} - \frac{p}{\Omega^2 - 2}\cos\xi, \tag{6}$$

where A and B are constants determined by the initial conditions. Since $\Omega \gg 1$, the sum of the first three terms in equation (6) is a slowly varying function of ξ and the last term can be expressed approximately as $p/\Omega^2 \cos\xi$.

In the following, we shall assume that the solution of the nonlinear equation (5) with sufficiently large Ω can be expressed as

$$y = \phi(\xi) - k\cos\xi, \tag{7}$$

where $\phi(\xi)$ is a slowly varying function of ξ and $k = p/\Omega^2$. Equation (7) indicates that the solution of equation (5) can be obtained by the superposition of a sinusoidal function of ξ and *a* slowly varying function of ξ. This assumption will be justified by the numerical solution of equation (5). We shall discuss this point later. From equations (5) and (7), we obtain

$$\ddot{\phi} + \frac{1}{\Omega^2}[(\phi - k\cos\xi)^3 - 3(\phi - k\cos\xi)^2 + 2(\phi - k\cos\xi) - p_0] = 0. \tag{8}$$

In the following, we shall use the cycle-averaging technique [8] for the investigation of the dynamic buckling of the truss. Multiplying equation (8) by $1/2\pi\, d\xi$ and integrating from ξ to $\xi + 2\pi$, we obtain

$$\ddot{\bar{\phi}} + \frac{1}{2\Omega^2}[(\bar{\phi} - 1)(2\bar{\phi}^2 - 4\bar{\phi} + 3k^2) - 2p_0] = 0, \tag{9}$$

where $\bar{\phi}$ is the cycle-averaged value of ϕ. Note that $\bar{\phi}$ is also a slowly varying function of ξ.

The intial conditions are

$$y(0) = y_0 \tag{10}$$

and

$$\dot{y}(0) = 0. \tag{11}$$

Hence,

$$\bar{\phi}(0) = y_0 + k = \phi_0 \tag{12}$$

and

$$\dot{\bar{\phi}}(0) = 0. \tag{13}$$

Equations (9), (12) and (13) define an autonomous system. Dynamic buckling of the simple truss can be investigated by the stability analysis of the autonomous system.

The energy equation can be derived from equation (9) by integration, using the initial conditions, equations (12) and (13). It is

$$K+U = 0, \tag{14}$$

where

$$K = \tfrac{1}{2}\dot{\bar{\phi}}^2 \tag{15}$$

and

$$U = \frac{1}{2\Omega^2}[\tfrac{1}{2}(\bar{\phi}^4-\phi_0^4)-2(\bar{\phi}^3-\phi_0^3)+\tfrac{1}{2}(3k^2+4)(\bar{\phi}^2-\phi_0^2)-(3k^2+2p_0)(\bar{\phi}-\phi_0)]. \tag{16}$$

In equations (15) and (16), K is the kinetic energy and U is the potential energy with its base $U = 0$ at $\bar{\phi} = \phi_0$. At the moment of extreme cycle-averaged deformation, $\dot{\bar{\phi}} = 0$. Thus, by equation (14), we can determine the extreme value of $\bar{\phi}$ which is denoted by ϕ_m. It is found that ϕ_m satisfies the equation

$$\phi_m^3+(\phi_0-4)\phi_m^2+(\phi_0^2-4\phi_0+3k^2+4)\phi_m+\phi_0^3-4\phi_0^2+(3k^2+4)\phi_0-2(3k^2+2p_0) = 0. \tag{17}$$

In the following, we shall study the case $y_0 = 0$. Thus, we have $\phi_0 = k$ and equation (17) is reduced to

$$\phi_m^3+(k-4)\phi_m^2+4(k^2-k+1)\phi_m+2(2k^3-5k^2+2k-2p_0) = 0. \tag{18}$$

For given p_0 and k, ϕ_m can be solved numerically from equation (18). Dynamic buckling can then be investigated by the k vs. ϕ_m curve. Let us first consider the following special case.

A SPECIAL CASE—SIMPLE TRUSS UNDER PURE OSCILLATORY LOADING

In this special case, $P_0 = 0$. Hence $p_0 = 0$ and equation (18) can be reduced to

$$(\phi_m-k)(\phi_m+k-2)(\phi_m^2-2\phi_m-2k+4k^2) = 0. \tag{19}$$

Equation (19) defines two straight lines and one elliptical curve as shown in Fig. 2. When the value of k increases from zero, the value of ϕ_m varies along the path $0ABCDE$. Note that at the turning points A and D, the variation of ϕ_m with respect to k changes suddenly from one pattern to another. At the local maximum point B, the value of ϕ_m jumps discontinuously. We shall refer to the points A and D as the points of dynamic bifurcation and the point B as the point of dynamic snap-through. It can be easily shown that $k = \phi_m = 4/5$ at A; $k = (1+5^{1/2})/4$ and $\phi_m = 1$ at B; $k = (1+5^{1/2})/4$ and $\phi_m = (7-5^{1/2})/4$ at C and $k = \phi_m = 1$ at D.

In order to check the assumption of equation (17), we may solve equation (5) numerically with $p_0 = y(0) = \dot{y}(0) = 0$. It can be done by the Runge–Kutta method where the value of Ω is chosen as 50. From the numerical solution of $y(\xi)$, we can determine the cycle-averaged value $\bar{\phi}(\xi)$ which is plotted against ξ in Fig. 3. Next, we can obtain the extreme cycle-averaged values ϕ_m which are given in Table 1. The relation of k and ϕ_m determined by

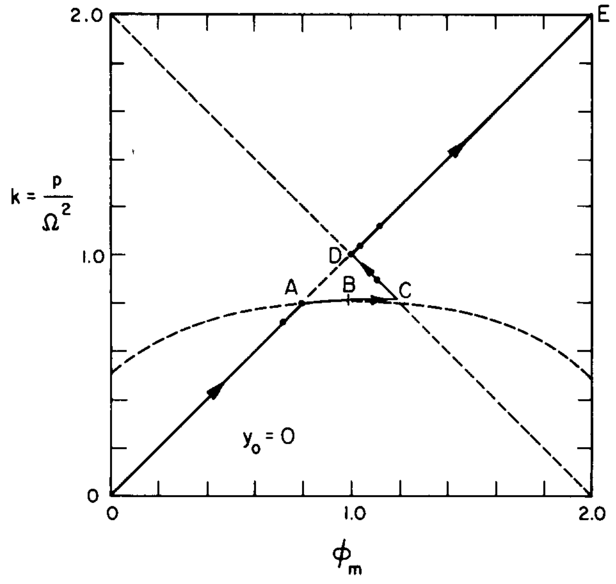

FIG. 2. k–ϕ_m curve for $p_0 = 0$.

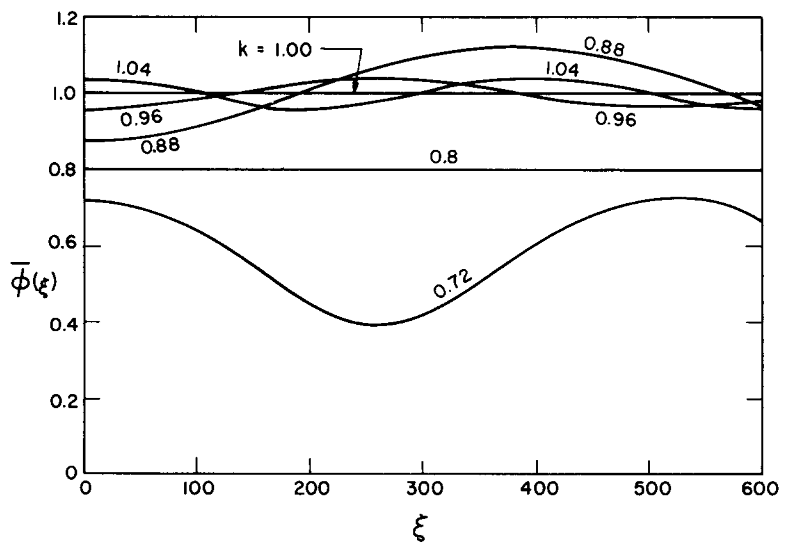

FIG. 3. Response curves $\bar{\phi}(\xi)$ vs. ξ for various values of p/Ω^2.

TABLE 1. ϕ_m–k RELATION DETERMINED FROM THE NUMERICAL SOLUTION OF EQUATION (5)

k	0·72	0·80	0·88	0·96	1·00	1·04	1·12
ϕ_m	0·72	0·80	1·12	1·04	1·00	1·04	1·12

the numerical solution is also shown in Fig. 2 by dark circles. It is seen that these points are just on the curve $0ABCDE$ as predicted theoretically. Hence, the assumption of equation (7) is justified.

It would be interesting to study the bifurcation and snap-through phenomena from $\bar{\phi}(\xi)$ curves in Fig. 3. It is found that at the points of dynamic bifurcation A and D and the point of dynamic snap-through B in Fig. 2, the value of $\bar{\phi}$ becomes independent of ξ. This behavior can be proved easily by substituting the values of k and ϕ_m at A, D and B in Fig. 2 into the equation of motion, equation (9), and the energy equation, equation (14). From these substitutions, we conclude that $\dot{\bar{\phi}} = \ddot{\bar{\phi}} = 0$ at both the points of dynamic bifurcation and the point of dynamic snap-through. Hence, the corresponding $\bar{\phi}(\xi)$ curve remains horizontal in Fig. 3.

THE GENERAL CASE OF THE TRUSS PROBLEM

In the general case, equation (18) is solved numerically by the Newton–Raphson iterative process for different values of p_0. The results are shown in Fig. 4. The following conclusions on the dynamic buckling of a shallow truss can be drawn:

1. By using the assumption $p \gg p_0$, we may conclude that when p is small, ϕ_m is proportional to p. When p reaches a certain critical value, the pattern of the $\phi(\xi)$ curve changes suddenly and dynamic bifurcation occurs.

2. For continuous increment of p, the extreme value of the cycle-averaged deflection jumps discontinuously and dynamic snap-through occurs.

3. The critical value of p for either dynamic bifurcation or dynamic snap-through is inversely proportional to the square of the frequency and decreases with increasing mean load p_0.

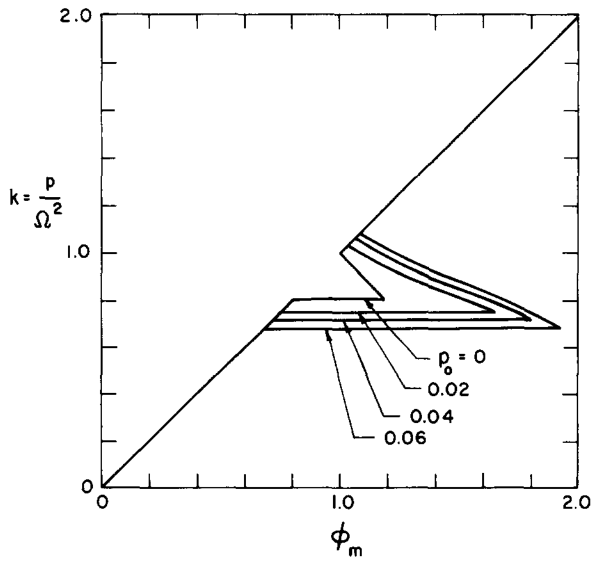

FIG. 4. k–ϕ_m curves for different p_0.

4. The interval of the jump during the dynamic snap-through increases with the mean load p_0.

DYNAMIC BUCKLING OF A SINUSOIDAL SHALLOW ARCH

An elastic shallow arch of span L, cross-sectional area A, moment of inertia of the cross section about the neutral axis I, mass per unit length ρ and initial shape $w_0(x)$ is under the action of a dynamic load $q(x, t)$ applied at zero time as shown in Fig. 5. The deformed shape of the arch at any time t is denoted by $w(x, t)$. The equation of motion of the arch is given in [3] as

$$EI(w'''' - w_0'''') + w'' \frac{A}{2L} E \int_0^L [(w_0')^2 - (w')^2]\,dx + q + \rho \ddot{w} = 0, \tag{20}$$

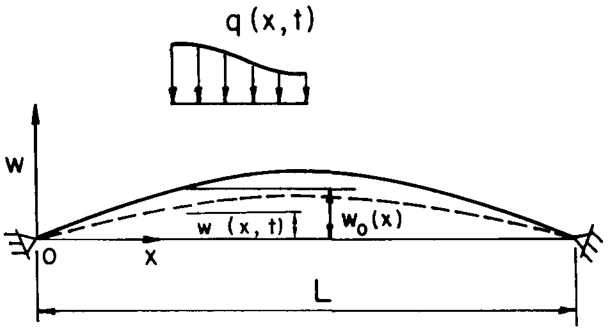

FIG. 5. Geometry of an elastic shallow arch.

where prime represents the partial differentiation with respect to the space coordinate x and dot the partial differentiation with respect to t. Since the ends of the arch are supported by hinges, the boundary conditions are

$$w_0(0) = w_0(L) = w(0, t) = w(L, t) = 0 \tag{21}$$

and

$$w''(0, t) = w_0''(0); \qquad w''(L, t) = w_0''(L). \tag{22}$$

We shall assume that the space-wise load distribution is sinusoidal in the first harmonic and consider only the first and second harmonics for the initial and the deformed shapes of the arch. Hence, we have

$$w_0(x) = 2\left(\frac{I}{A}\right)^{1/2} \left(y_0 \sin \frac{\pi x}{L} + z_0 \sin \frac{2\pi x}{L}\right), \tag{23}$$

$$w(x, t) = 2\left(\frac{I}{A}\right)^{1/2} \left[y(t) \sin \frac{\pi x}{L} + z(t) \sin \frac{2\pi x}{L}\right], \tag{24}$$

and

$$q(x) = \frac{2\pi^4 EI}{L^4}\left(\frac{I}{A}\right)^{1/2} p \sin\frac{\pi x}{L}. \tag{25}$$

Denote

$$\bar{\tau} = \frac{\pi^2}{L^2}\left(\frac{EI}{\rho}\right)^{1/2} t. \tag{26}$$

Equation (20) can be written as

$$\ddot{y}+y-y_0+y(y^2-y_0^2+4z^2-4z_0^2)+p = 0 \tag{27}$$

and

$$\ddot{z}+16(z-z_0)+4z(y^2-y_0^2+4z^2-4z_0^2) = 0 \tag{28}$$

where dot represents the differentiation with respect to $\bar{\tau}$.

If the initial shape is approximately the first harmonic function of x with a small geometrical imperfection in a form of the second function of x, then $z_0 = 0^+$ and equations (27) and (28) become

$$\ddot{y}+y-y_0+y(y^2-y_0^2+4z^2)+p = 0 \tag{29}$$

and

$$\ddot{z}+16z+4z(y^2-y_0^2+4z^2) = 0. \tag{30}$$

Since the load is applied at $t = 0$ instantly, we have the following initial conditions for a sinusoidal arch with small initial imperfections with a shape of a second harmonic function of x:

$$y(0) = y_0, \qquad z(0) = 0^+, \qquad y(0) = 0, \qquad \dot{z}(0) = 0. \tag{31}$$

In our problem, it is assumed that the applied load is periodic in time with a large dimensionless frequency Ω. Hence,

$$p = r_0+r\cos\Omega\bar{\tau}, \tag{32}$$

where r_0 is the mean value of the applied load which can be considered as a step-function of time and r is the amplitude of the oscillatory load. In the following, we shall assume that $r \gg r_0$. Denote $\tau = \omega\bar{\tau}$ and $(\dot{\ }) = \partial/\partial\tau(\ )$. We have the following governing equations of motion:

$$\ddot{y}+\frac{1}{\Omega^2}[y-y_0+y(y^2-y_0^2+4z^2)+r_0+r\cos\tau] = 0 \tag{33}$$

and

$$\ddot{z}+\frac{1}{\Omega^2}[16z+4z(y^2-y_0^2+4z^2)] = 0. \tag{34}$$

From the solution of the linearized equations of (33) and (34), we may assume that, for sufficiently large value of ω, the solution of equations (33) and (34) can be expressed approximately as

$$y = \varphi+k\cos\tau \tag{35}$$

and

$$z = \psi, \tag{36}$$

where $k = r/\Omega^2$ and $\phi(\tau)$ and $\psi(\tau)$ are slowly varying functions of τ. Substituting equations (35) and (36) into equations (33) and (34) and applying the cycle-averaging process, we obtain the following coupled nonlinear differential equations:

$$\ddot{\bar{\phi}} + \frac{1}{\Omega^2}[\bar{\phi} - y_0 + r_0 + \bar{\phi}(\bar{\phi}^2 - y_0^2 + 4\bar{\psi}^2) + \tfrac{3}{2}k^2\bar{\varphi}] = 0 \tag{37}$$

and

$$\ddot{\bar{\psi}} + \frac{2}{\Omega^2}\bar{\psi}[8 + 2(\bar{\phi}^2 - y_0^2 + 4\bar{\psi}^2) + k^2] = 0, \tag{38}$$

where $\bar{\phi}$ and $\bar{\psi}$ are the cycle-averaged values of ϕ and ψ, respectively. The initial values of ϕ and ψ can be found from equations (35) and (36). They are

$$\phi(0) = y_0 - k = \phi_0 \tag{39}$$

$$\psi(0) = 0 \tag{40}$$

and

$$\dot{\phi}(0) = \dot{\psi}(0) = 0. \tag{41}$$

Hence, we have

$$\bar{\phi}(0) = \phi_0 \tag{42}$$

$$\bar{\psi}(0) = 0 \tag{43}$$

and

$$\dot{\bar{\phi}}(0) = \dot{\bar{\psi}}(0) = 0. \tag{44}$$

Equations (37), (38) and (42)–(44) define an autonomous system. To investigate the stability of the system, let us first find the equilibrium points of the system which can be derived from equations (37) and (38) by setting $\ddot{\bar{\phi}} = \ddot{\bar{\psi}} = 0$. Denote the coordinates of the equilibrium points by $(\bar{\phi}_e, \bar{\psi}_e)$. It is found that there are five equilibrium points. The coordinate of three equilibrium points satisfy

$$\bar{\varphi}_e^3 + (1 - y_0^2 + \tfrac{3}{2}k^2)\bar{\phi}_e + r_0 - y_0 = 0 \tag{45}$$

and

$$\bar{\psi}_e = 0, \tag{46}$$

and the coordinates of the other two equilibrium points are

$$\bar{\phi}_e = \frac{y_0 - r_0}{k^2 - 3} \tag{47}$$

and

$$\bar{\psi}_e = \pm\left\{\frac{1}{4}\left[y_0^2 - \left(\frac{y_0 - r_0}{k^2 - 3}\right)^2\right] - \frac{k^2}{8} - 1\right\}^{1/2}. \tag{48}$$

In the following, we shall utilize the method of potential energy surface [1, 3] to investigate the upper and lower bounds of the critical valve of k for dynamic instability. The potential energy U of the system can be obtained easily by the integration of equations (37) and (38), using the initial conditions equations (42)–(44). It is

$$U = \frac{1}{\Omega^2}[\tfrac{1}{2}(\bar{\phi}-\phi_0)^2 + 8\bar{\psi}^2 + \tfrac{1}{4}(\bar{\phi}^2 - \phi_0^2 + 4\bar{\psi}^2)^2 - (r_0 - k)(\bar{\phi}-\phi_0)$$

$$+\tfrac{1}{2}(\bar{\phi}^2 - y_0^2 + \tfrac{3}{2}k^2)(\bar{\phi}^2 - \phi_0^2 + 4\bar{\psi}^2) - 2k^2\bar{\psi}^2]. \tag{49}$$

Considering the value of U as the height, we can construct a surface above the $\bar{\phi}$–$\bar{\psi}$ plane. This surface is referred to as the potential surface. Note that $U = 0$ at the initial point $(\phi_0, 0)$. At the equilibrium points defined by equations (45) and (46), the potential energy has either maximum or minimum values. Hence, these equilibrium points correspond to either local maximum or minimum points on the potential surface. There are two stable equilibrium points at two minimum points of the potential surface and an unstable equilibrium point at the maximum point of the potential surface. The equilibrium points defined by equations (47) and (48) are two saddle points on the potential surface.

During the motion of the arch, the point of trajectory moves on the potential surface in a manner similar to a small ball rolling on a frictionless surface. For given values of y_0 and r_0, when the value of k is small, the region of the motion of the ball is restricted to the neighborhood of the minimum point close to the initial point on the potential surface. However, when the value of k is sufficiently large, the trajectory may reach the neighborhood of the far minimum point and dynamic snap-through is thus introduced. Since there is a hill between two dips on the potential surface, the far minimum point would be reached in the trajectory if the elevation of the maximum point is equal to or less than the elevation of the initial point which is zero. Hence, by setting the potential energy zero at the maximum point, we can obtain an upper bound of the critical value of k for dynamic snap-through. Note that the far minimum point can also be reached by a trajectory passing through the neighborhood of the saddle point. Therefore, the lower bound of the critical value of k for dynamic snap-through can be derived by setting the potential energy zero at the saddle point.

The upper and lower bounds of the critical values of k for given y_0 and r_0 can be found numerically from equations (45)–(49), using a cut-and-try technique. For any value of k, the locations of equilibrium points are first found from equations (45)–(48) and the potential energy is then calculated according to equations (49). The upper and lower bounds of the critical value of k are determined from the conditions of zero potentials at unstable equilibrium points. Numerical computations are carried out for $y_0 = 2$, 3, 4 and 5 and $r_0 = 2$, 4, 6, 8, 10 and 12. The results are given in Table 2 and also shown in Fig. 6, where k_u is the upper bound of the critical value of k and k_l is the lower bound of the critical value of k. It is found that the lower bound exists only within a certain range of r_0. When r_0 is within the range, for any given y_0, k_l decreases with increasing r_0 as we may predict. When r_0 is not within the range, the saddle point would vanish before its potential becomes zero. In this case, there is no lower bound for the critical value of k. It is also found that the upper bound does not exist for $r = 4$, 6, 8 and 10. In this case, the maximum point on the potential surface would disappear before its potential becomes zero. When $y_0 = 5$ and $r_0 = 12$, the upper bound of the critical value of k is $k_u = 2{\cdot}874$. From Fig. 6, it is also noted that when

TABLE 2. UPPER AND LOWER BOUNDS OF THE CRITICAL VALUES OF k FOR DYNAMIC SNAP-THROUGH

y_0	r_0	k_u	k_l
2·0	2·0	1·366	–
3·0	4·0	–	1·332
3·0	6·0	–	1·058
3·0	7·0	–	0·944
4·0	4·0	–	1·350
4·0	6·0	–	1·134
4·0	8·0		0·958
4·0	10·0	–	0·799
4·0	12·0	–	0·649
5·0	4·0	–	1·361
5·0	6·0	–	1·182
5·0	8·0	–	1·034
5·0	10·0	–	0·901
5·0	12·0	2·874	0·776

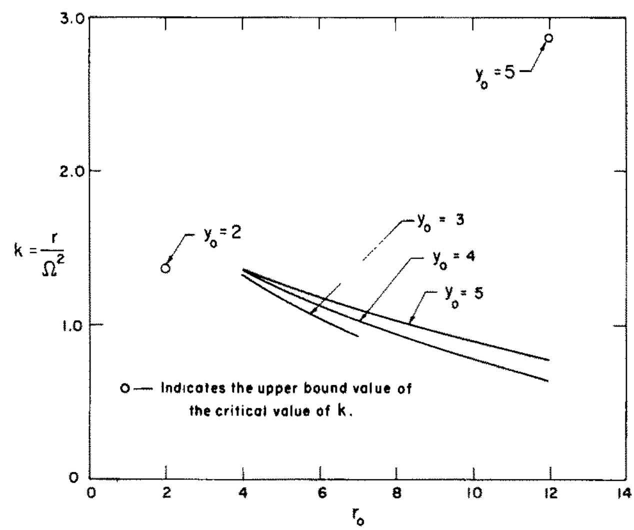

FIG. 6. Lower bound of critical value of k for dynamic snap-through.

$y_0 = 2$, the lower bound does not exist and the upper bound exists at $r_0 = 2$. In the latter case, $k_u = 1{\cdot}366$ is the critical value of k for dynamic snap-through.

Although in this paper we are dealing with elastic arches, the lower bound of the critical load found here can also be considered as the lower bound of the critical load for arches with dissipation. Investigations along this line are found in [2–4, 6] and is omitted in this paper.

CONCLUDING REMARKS

1. In this paper, an energy method incorporated with cycle-averaging technique is introduced for the investigation of stability of instationary systems.

2. The study of the simple truss problem indicates that the cycle-averaging technique is good for the analysis of dynamic stability of structures provided the frequency of the load is sufficiently high.

3. The paper provides bounds of the critical load for dynamic buckling of elastic shallow imperfect arches subject to periodic loading.

Acknowledgement—The author wishes to acknowledge the assistance of B. Vahidi in preparing Fig. 3.

REFERENCES

[1] N. J. HOFF and V. G. BRUCE, Dynamic analysis of the buckling of laterally loaded flat arches. *J. Math. Phys.* **32**, 276–288 (1954)
[2] W. NACHBAR and N. C. HUANG, Dynamic snap-through of a simple viscoelastic truss. *Q. appl. Math.* **25**, 65–82 (1967).
[3] N. C. HUANG and W. NACHBAR, Dynamic snap-through of imperfect viscoelastic shallow arches. *J. appl. Mech.* **35**, 289–296 (1968).
[4] M. H. LOCK, Snapping of a shallow sinusoidal arch under a step pressure load. *AIAA Jnl* **4**, 1249–1256 (1966).
[5] C. S. HSU, Stability of shallow arches against snap-through under time-wise step loads. *J. appl. Mech.* **33**, 31–39 (1968).
[6] M. C. CHEUNG and C. D. BABCOCK, JR., Stability of a clamped shallow arch subjected to impulsive loading. *AIAA Jnl* **8**, 1434–1439 (1970).
[7] E. METTLER, Stability and Vibration Problems of Mechanical Systems Under Harmonic Excitation, *Dynamic Stability of Structure*, edited by G. HERRMANN, 169–188, Pergamon Press (1967).
[8] N. MINORSKY, *Nonlinear Oscillations*. Van Nostrand (1962).

(*Received* 14 *May* 1971; *revised* 16 *August* 1971)

Абстракт—На основе двух простых моделей, исследуется динамическое выпучивание упругих пологих конструкций, подверженнных действию периодической нагрузки. Когда частота и величина колебательной нагрузки достаточно высокие, тогда можно применить метод усреднения циклов. с целью определения независимой системы для циклически усредненного движения конструкции Используется, затем, энергический метод для определения верхнего и нижнего пределов критической нагрузки для случая динамического выпчивания.